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Strong convergence results for a new class of Jungck-strictly pseudo-contractive mappings via inertial and accelerated iterative schemes

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ABSTRACT

In this paper, we introduce a new class of Jungck-strictly pseudo-contractive mappings in real Hilbert spaces and investigate its relationship with the class of Jungck generalised pseudo-contractive mappings. We prove that every Jungck-strictly pseudo-contractive mapping is Jungck generalised pseudo-contractive, whereas the converse fails, thereby establishing a proper subclass inclusion. Motivated by this operator class, we study three iterative approximation methods: a Jungck–Halpern iteration, an inertial Jungck–Halpern scheme, and an accelerated two-step Jungck iteration. Under suitable assumptions on the mappings and control parameters, we establish strong convergence of the generated sequences to coincidence points. For the representative linear example, the inertial Jungck–Halpern and Jungck–Halpern schemes attained the residual tolerance 10^{-6} after 111 and 112 updates, respectively; the accelerated two-step scheme required 195 updates, whereas the Jungck–Mann comparison scheme did not attain the tolerance within 10,000 updates. The numerical experiment was generated directly from the displayed iterative formulas and provides reproducible evidence of convergence without asserting that any scheme is universally fastest for all admissible parameter choices. The results extend existing convergence theory for Jungck-type operators and provide implementable frameworks for coincidence-point approximation in real Hilbert spaces.

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Introduction

Fixed point theory is a fundamental tool in nonlinear analysis, providing a unified framework for studying the existence and approximation of solutions to a wide range of problems arising in differential equations, optimisation, and variational inequality theory. Since Banach's pioneering work on contraction mappings [1], numerous generalisations have been introduced to accommodate broader classes of nonlinear operators in metric and Hilbert spaces while preserving the convergence properties of associated iterative schemes. Interested readers may refer to [2–9] and references therein.

In particular, nonexpansive and pseudo-contractive mappings have attracted significant attention because of their close connections with variational inequalities and related problems. Strictly pseudo-contractive mappings in Hilbert spaces arose as a natural generalisation of nonexpansive mappings and have been developed extensively by many authors (see, for example, [10–13, 21- 22]). These mappings are closely linked to monotone operator theory and

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variational inequalities, making them suitable for studying iterative schemes such as the Halpern [5] and Mann [7] iterations.

Jungck introduced commuting mappings and coincidence-point theory [14, 15], which subsequently developed into what are now known as Jungck-type iterative schemes. These schemes are effective for studying common fixed points of pairs of mappings without requiring strict commutativity.

More recently, several authors have investigated Jungck-type extensions of contractive and pseudo-contractive mappings, leading to the development of Jungck generalised pseudo-contractive mappings and associated iterative processes (see [16-20]). In particular, Olatinwo and Omidire [20] defined Jungck generalised pseudo-contraction as follows:

Definition 1.1 [20]: Let H be a real Hilbert space. An operator $T: H \to H$ is said to be Jungck generalised pseudo-contractive, or a Jungck generalised pseudo-contraction if there exist $r > 0$ and an operator $S: H \to H$ such that $\forall u, v \in H$,
$\| Tu - Tv \|^2 \leq r^2 \| Su - Sv \|^2 + \| Tu - Tv - r(Su - Sv) \|^2, T(H) \subset S(H)$.

Despite these advances, most existing results rely on either: (i) classical Mann-type iterations, or (ii) standard Jungck hybrid schemes without acceleration mechanisms. Moreover, the convergence analysis typically requires relatively strong assumptions or does not incorporate modern acceleration techniques such as inertial or two-step correction processes.

Motivated by these observations, the present paper introduces a new class of Jungck-strictly pseudo-contractive mappings in real Hilbert spaces. This class forms a proper subclass of the Jungck generalised pseudo-contractive mappings. These studies combine the structural flexibility of Jungck-type operators with the convergence properties of pseudo-contractive mappings in Hilbert spaces.

The main contributions of this work are summarised as follows:

- A new Jungck-strictly pseudo-contractive operator class is introduced, and its structural properties are investigated.
- A proper subclass relationship with Jungck generalised pseudo-contractive mappings is established.
- Strong convergence theorems are proved using Jungck–Halpern iterations under mild monotonicity-type conditions.
- An inertial Jungck–Halpern scheme is proposed and shown to converge strongly to a coincidence point.
- An accelerated two-step Jungck iteration is introduced, providing improved convergence behaviour over classical Jungck–Mann methods.

The results obtained extend and unify several existing convergence theorems in the literature and provide new iterative frameworks for solving fixed point and coincidence problems in Hilbert spaces.

Materials and methods

We begin with several technical lemmas used in the convergence analysis of the proposed iterative schemes.

Lemma 2.1 (Hilbert space identity). *Let H be a real Hilbert space.*

Then for all $x, y \in H$ and $\lambda \in [0,1]$
$\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2$.

Lemma 2.2 (Fejér monotonicity criterion) [6]: *Let $C \subset H$ be nonempty and let $\{x_n\} \subset H$. If* $\| x_{n+1} - p \|^2 \leq \| x_n - p \|^2 + \epsilon_n, \quad \forall p \in C,$
where $\epsilon_n \geq 0$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$, then $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| x_n - p \|$ exists for every $p \in C$.

Lemma 2.3 (Demiclosedness principle [2,6]) *Let H be a real Hilbert space and let $T: H \to H$ be such that $I - T$ is demiclosed at zero. If* $x_n \rightharpoonup x$ and $\| x_n - Tx_n \| \to 0$, then $x \in \mathrm{Fix}(T)$.

Lemma 2.4 (Weak cluster point principle [2,4,6]):
Let H be a real Hilbert space and let $T: H \to H$ be such that $I - T$ is demiclosed at zero.
Suppose $\{x_n\} \subset H$ satisfies $\| x_n - Tx_n \| \to 0$.
If $x_{n_k} \rightharpoonup x$ for some subsequence $\{x_{n_k}\}$,
then $x \in \mathrm{Fix}(T)$.

Consequently, every weak cluster point of $\{x_n\}$ belongs to $\mathrm{Fix}(T)$.

Lemma 2.5 (Xu-type summability lemma [8]) *Let $\{a_n\}, \{\alpha_n\} and \{\sigma_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n) a_n + \alpha_n \gamma_n + \sigma_n, n \geq 0$$

where $\alpha_n \in (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$,

$$\sigma_n = \varepsilon_n \geq 0, \qquad \sum_{n=1}^{\infty} \sigma_n < \infty \quad \textit{and}$$

$$\limsup_{n\to\infty} \gamma_n \leq 0.$$

Then, $a_n \to 0$.

Definition 2.6 (Coincidence point [14,15]): *Let D be a nonempty set and let $S, T: D \to D$ be two self-*

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mappings. A point $x \in D$ *is called a coincidence point of the pair* (S,T) *if* $Sx = Tx$.

The set of all coincidence points of S and T is denoted by

$$C(S,T) = \{x \in D: Sx = Tx\}.$$

The common value $w = Sx = Tx$ is called a *point of coincidence* of the pair (S,T).

Remark 2.7 *If* $S = I$*, the identity mapping on* D*, then coincidence points of* (I,T) *reduce to fixed points of* T*, since* $Ix = Tx \Leftrightarrow x = Tx$.

Thus, coincidence point theory extends the classical fixed point framework.

Results

Preliminary results

Definition 3.1.1 *Let* D *be a real Hilbert space and let* $T,S: D \to D$ *be self-mappings. Then* T *is called a Jungck-strictly pseudo-contractive operator with respect to* S *if there exists a constant* $L \in (0,1)$ *such that for all* $u, v \in D$,

$$\| Tu - Tv \|^2 \leq \| Su - Sv \|^2 + L \| (S-T)u - (S-T)v \|^2. \quad (3.1)$$

Remark 3.1.2 *If* $S = I$*, where* I *denotes the identity operator on* D*, then the above definition reduces to the classical strictly pseudo-contractive operator.*

Relationship with Jungck generalised pseudo-contractive mappings

In this subsection, we investigate the relationship between the class of Jungck-strictly pseudo-contractive mappings introduced in Definition 3.1.1 and the existing class of Jungck generalised pseudo-contractive mappings of Olatinwo et al. [20] defined in Definition 1.1 above. The aim is to determine whether the newly introduced class is equivalent to, stronger than, or independent of the known Jungck pseudo-contractive framework.

We compare (3.1) with the standard Jungck generalised pseudo-contractive condition, namely,

$$\langle Tu - Tv, Su - Sv \rangle \leq r \| Su - Sv \|^2, r \geq 1, u, v \in D. \quad (3.2)$$

The following result establishes a fundamental implication.

Proposition 3.2.1 *Let* D *be a real Hilbert space and let* $S,T: D \to D$ *be self-mappings. If* T *is Jungck-strictly pseudo-contractive with respect to* S*, then* T *satisfies a Jungck generalised pseudo-contractive inequality.*

Proof: Assume that T satisfies (3.1). Let

$$Sd := Su - Sv, \qquad Td := Tu - Tv.$$

Then (3.1) becomes

$$\| Td \|^2 \leq \| Sd \|^2 + L \| Sd - Td \|^2. \quad (3.3)$$

Expanding the last term gives

$$\| Sd - Td \|^2 = \| Sd \|^2 + \| Td \|^2 - 2\langle Sd, Td \rangle. \quad (3.4)$$

Substituting this into the inequality (3.3) yields

$$\| Td \|^2 \leq \| Sd \|^2 + L(\| Sd \|^2 + \| Td \|^2 - 2\langle Sd, Td \rangle).$$

Rearranging terms, we obtain

$$(1-L) \| Td \|^2 \leq (1+L) \| Sd \|^2 - 2L\langle Sd, Td \rangle.$$

Hence,

$$2L\langle Sd, Td \rangle \leq (1+L) \| Sd \|^2 - (1-L) \| Td \|^2.$$

Since

$$(1-L) \| Td \|^2 \geq 0,$$

it follows that

$$2L\langle Sd, Td \rangle \leq (1+L) \| Sd \|^2.$$

Consequentl

$$\langle Sd, Td \rangle \leq \frac{1+L}{2L} \| Sd \|^2.$$

Therefore

$$\langle Tu - Tv, Su - Sv \rangle \leq r \| Su - Sv \|^2,$$

where

$$r = \frac{1+L}{2L}.$$

This proves that every Jungck-strictly pseudo-contractive mapping satisfies a Jungck generalised pseudo-contractive inequality. The converse implication does not generally hold.

Proposition 3.2.2. The class of Jungck generalised *pseudo-contractive mappings is not contained in the class of Jungck-strictly pseudo-contractive mappings.*

Proof: Consider the Hilbert space $D = \mathbb{R}$ endowed with the usual norm and define

$$Sx = x, \qquad Tx = 2x.$$

Then for arbitrary $x, y \in \mathbb{R}$,

$$Su - Sv = x - y, \qquad Tu - Tv = 2(x-y).$$

First, we verify the Jungck generalized pseudo-contractive condition. Observe that

$$\langle Tu - Tv, Su - Sv \rangle = 2(x-y)^2.$$

Choosing $r = 2$, we obtain

$$2(x-y)^2 \leq 2(x-y)^2,$$

so condition (3.2) holds.

Next, we examine (3.1). Substituting the above mappings into (3.1), we obtain

$$4(x-y)^2 \leq (x-y)^2 + L(x-y)^2.$$

Equivalently

$$4 \leq 1 + L. \quad (3.5)$$

However, since $0 \leq L < 1$,

we have $1 + L < 2$,

which contradicts the inequality (3.5). Hence, 3.1 fails.

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Therefore, a Jungck generalised pseudo-contractive mapping need not be Jungck-strictly pseudo-contractive.

The preceding propositions yield the following characterisation.

Theorem 3.2.3 The class of Jungck-strictly pseudo-contractive mappings forms a proper subclass of the class of Jungck generalised pseudo-contractive mappings.

Proof: By Proposition 3.2.1, every Jungck-strictly pseudo-contractive mapping satisfies a Jungck generalised pseudo-contractive inequality. Proposition 3.2.2 establishes that the converse implication fails. Therefore, the inclusion is proper.

Remark 3.2.4 The above result shows that Definition 3.1.1 is not merely a reformulation of the existing Jungck generalised pseudo-contractive framework. Rather, it defines a strictly narrower operator class possessing stronger geometric restrictions. Consequently, convergence theorems established under Definition 3.1.1 are not immediate corollaries of the standard Jungck generalised pseudo-contractive theory and therefore provide genuinely new contributions to the literature.

Corollary 3.2.5 [Boundedness of Jungck-type iterates] Let $S, T: D \to D$ satisfy the Jungck-strictly pseudo-contractive condition and suppose $C(S,T) \neq \emptyset$.

Assume that the generated sequence $\{x_n\}$ satisfies

$$\| Sx_{n+1} - Sp \|^2 \leq \| Sx_n - Sp \|^2 + \epsilon_n,$$

for some $p \in C(S,T)$, where $\epsilon_n \geq 0$, $\sum_{n=0}^{\infty} \epsilon_n < \infty$.

Then $\{Sx_n\}$ is bounded. Moreover, if S^{-1} exists and maps bounded subsets of $S(D)$ into bounded subsets of D, then $\{x_n\}$ is bounded.

Proof: Let $a_n := \| Sx_n - Sp \|^2$.

Then

$$a_{n+1} \leq a_n + \epsilon_n.$$

By induction

$$a_n \leq a_0 + \sum_{k=0}^{n-1} \epsilon_k.$$

Since

$$\sum_{k=0}^{\infty} \epsilon_k < \infty$$

it follows that

$$a_n \leq a_0 + \sum_{k=0}^{\infty} \epsilon_k < \infty.$$

Hence $\{a_n\}$ is bounded, and therefore $\{Sx_n\}$ is bounded.

If S^{-1} maps bounded subsets of $S(D)$ into bounded subsets of D, the boundedness of $\{x_n\} = S^{-1}(Sx_n)$ follows immediately.

Since S(D) is a closed convex subset of H, the bounded sequence $\{Sx_n\}$ remains entirely in S(D), and every weak cluster point also belongs to S(D). Hence all limit points lie in the domain of the inverse mapping S^{-1}.

The following elementary estimate controls the inertial perturbation arising in the proposed Jungck inertial scheme (cf. [23, 24]).

Lemma 3.2.6 (Control of inertial term). *Let* $\{x_n\}$ *be generated by the inertial Jungck scheme*

$$Sy_n = Sx_n + \beta_n(Sx_n - Sx_{n-1}),$$

where

$$\{\beta_n\} \subset [0, \beta), \qquad \beta < 1,$$

and suppose

$$\sum_{n=1}^{\infty} \beta_n \| Sx_n - Sx_{n-1} \| < \infty.$$

Then

$\| Sy_n - Sx_n \| \to 0.$

Proof: From the inertial definition,

$$Sy_n - Sx_n = \beta_n(Sx_n - Sx_{n-1}).$$

Hence

$$\| Sy_n - Sx_n \| = \beta_n \| Sx_n - Sx_{n-1} \|.$$

Since

$$\sum_{n=1}^{\infty} \beta_n \| Sx_n - Sx_{n-1} \| < \infty,$$

the terms converge to zero.

Therefore, $\| Sy_n - Sx_n \| \to 0.$

Main results

Strong convergence results for Jungck–strictly Pseudo-contractive mappings

In this section, we establish strong convergence of the Jungck–Halpern iteration for the class of Jungck-strictly pseudo-contractive mappings. The proof is based on quasi-Fejér monotonicity, asymptotic regularity, and the demiclosedness principle.

Assumption 3.3.0: Let D be a nonempty closed convex subset of a real Hilbert space H. Throughout this section, assume that $S: D \to H$

is injective and satisfies that S(D) is a nonempty, closed and convex subset of H.

Consequently,

(1.) Every convex combination of elements of S(D) again belongs to S(D);

(2.) The inverse mapping

$$S^{-1}: S(D) \to D$$

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is well defined;

(3.) Every iterative scheme introduced below is well defined.

Remark

The assumption that S(D) is closed and convex is essential throughout the convergence analysis. Indeed, the Jungck–Halpern iteration is first constructed in the image space S(D), where each image iterate Sx_{n+1} is defined as a convex combination of points belonging to S(D). $T(D) \subseteq S(D)$ Since S(D) is convex, every convex combination appearing in the iteration also belongs to S(D). Consequently, the inverse mapping $S^{-1}: S(D) \to D$

is well defined and can be applied to recover the next iterate. Furthermore, the closedness of S(D) guarantees that weak and strong limit points arising in the convergence analysis remain in S(D), which is required for applying the demiclosedness principle and proving strong convergence.

Theorem 3.3.1 (Strong convergence of Jungck–Halpern iteration)

Let D be a real Hilbert space and let $S, T: D \to D$ be self-mappings satisfying:

(1) $T(D) \subseteq S(D)$, where $S(D)$ is a closed and convex subset of the Hilbert space H;

(2) S is injective and weakly continuous, and S(D) is a nonempty closed convex subset of H (see Assumption 3.3.0);

(3) $C(S,T) \neq \emptyset$;

(4) T is Jungck-strictly pseudo-contractive with respect to S, i.e.,

$$\| Tu - Tv \|^2 \leq \| Su - Sv \|^2 + L \| (S-T)u - (S-T)v \|^2, \quad L \in (0,1);$$

(5.) (Jungck monotonicity condition)

$$\backslash\langle Su - Sv, Tu - Tv \rangle \geq 0, \quad \forall u, v \in D.$$

Let $\{x_n\}$ be generated by the Jungck–Halpern iteration

$$Sx_{n+1} = \alpha_n Su + (1-\alpha_n)Tx_n;$$

(6.) (Control Sequence condition). The sequence$\{\alpha_n\} \subset (0,1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad and \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then there exists $p \in C(S,T)$ such that $Sx_n \to Sp$ strongly.

Moreover, if S^{-1} exists and is continuous on $S(D)$, then $x_n \to p$ strongly.

Proof:

Step 1: Boundedness of $\{Sx_n\}$

Let $p \in C(S,T)$ so that $Sp = Tp$. From the iteration,

$$Sx_{n+1} - Sp = \alpha_n(Su - Sp) + (1-\alpha_n)(Tx_n - Tp).$$

Since $Sx_{n+1} \in S(D)\ and\ Tx_n \in T(D) \subseteq S(D)$, and since S(D) is convex, by Assumption 3.3.0, the convex combination defining $Sx_{n+1} \in S(D)$. Consequently, the inverse mapping $S^{-1}: S(D) \to D$is applicable. Hence, the iterate x_{n+1} is well defined for every n. We split the resulting estimate into the following two inequalities.

Inequality A:

$$\| Sx_{n+1} - Sp \|^2 \leq (1-\alpha_n)\|Sx_n - Sp\|^2 + \alpha_n C_1$$
$$\| Sx_{n+1} - Sp \|^2 \leq \|Sx_n - Sp\|^2 + \alpha_n C_1$$

where $\alpha_n C_1 = \alpha_n \|Su - Sp\|^2$.

Now define $\epsilon_n := \alpha_n C_1$. Clearly,

$$\epsilon_n \geq 0, and \sum \varepsilon_n = C_1 \sum \alpha_n < \infty.$$

Therefore, the hypotheses of Lemma 2.2 are satisfied.

Step 2: Asymptotic regularity.

Inequality B: From Step 1,

$$\| Sx_{n+1} - Sp \|^2 \leq (1-\alpha_n) \| Sx_n - Sp \|^2 + \alpha_n C_1 - (1-\alpha_n)L \| Sx_n - Tx_n \|^2.$$

$Re\ a\ rranging,$

$$(1-\alpha_n)L \| Sx_n - Tx_n \|^2 \leq \| Sx_n - Sp \|^2 - \| Sx_{n+1} - Sp \|^2 + \alpha_n C_1,$$

$$\sum_{k=0}^{N} (1-\alpha_k)L \| Sx_k - Tx_k \|^2 \leq \| Sx_0 - Sp \|^2 - \| Sx_{N+1} - Sp \|^2 + C_1 \sum_{k=0}^{N} \alpha_k.$$

Now let $N \to \infty$.

Since $\sum \alpha_n < \infty$,

the right-hand side remains finite. Hence

$$\sum (1-\alpha_n) \| Sx_n - Tx_n \|^2 < \infty.$$

Since $1 - \alpha_n \geq c > 0$ for sufficiently large n, we conclude that

$$\| Sx_n - Tx_n \| \to 0.$$

Step 3: Weak cluster points.

Since $\{Sx_n\}$ is bounded (see Corollary 3.2.5), there exists a subsequence Sx_{n_k} and an element $u \in H$ such that

$$Sx_{n_k} \rightharpoonup u.$$

Because S(D) is closed and each $Sx_{n_k} \in S(D)$, we have $u \in S(D)$.

Consequently, u belongs to the domain of the inverse mapping S^{-1}.

Moreover, from Step 2, $\| Sx_{n_k} - Tx_{n_k} \| \to 0.$

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By the demiclosedness principle (Lemma 2.3), we conclude that $u \in C(S,T)$.

Hence there exists $p \in D$ such that $Sp = Tp$.

Step 4: Quasi-Fejér monotonicity.

From Step 1,
$\| Sx_{n+1} - Sp \|^2 \leq \| Sx_n - Sp \|^2 + \epsilon_n$, with $\epsilon_n \to 0$.

Thus $\{Sx_n\}$ is quasi-Fejér monotone with respect to $C(S,T)$. By Lemma 2.2, $\{Sx_n\}$ is bounded and $lim \| Sx_n - Sp \|$ exists. Since S(D) is closed, every weak cluster point of the bounded sequence $\{Sx_n\}$ remains in S(D). We use this observation in the strong convergence argument below.

Step 5: Strong convergence.

From Xu's lemma (Lemma 2.5),
$\| Sx_{n+1} - Sp \|^2 \leq (1 - \alpha_n) \| Sx_n - Sp \|^2 + \alpha_n C + \epsilon_n$,
with $\alpha_n \to 0$, $\sum \alpha_n = \infty$, and $\epsilon_n \to 0$.

Define $a_n \|Sx_n - Sp\|^2, \gamma_n = \alpha_n$ and $\sigma_n = \alpha_n C_1$.

Therefore, $a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n C + \sigma_n$.

Hence, $Sx_n \to Sp$.

Step 6: Strong convergence of x_n.

Since S^{-1} exists and is continuous on $S(D)$, $x_n = S^{-1}(Sx_n) \to S^{-1}(Sp) = p$.

Thus, $x_n \to p$.

Inertial Jungck–Halpern iterative scheme

In this section, we introduce an inertial term into the Jungck–Halpern framework and establish strong convergence in the Jungck-strictly pseudo-contractive setting. The inertial term uses information from previous iterates and may improve convergence performance.

Definition 3.4.1 (Inertial Jungck–Halpern iteration)

Let D be a real Hilbert space and let $S, T : D \to D$ be self-mappings. The inertial Jungck–Halpern iteration is defined by

$$Sy_n = Sx_n + \beta_n (Sx_n - Sx_{n-1}),$$
$$Sx_{n+1} = \alpha_n Su + (1 - \alpha_n) Ty_n, \qquad n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0, \beta)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 \leq \beta_n \leq \beta < 1$;
(iv) $\sum_{n=1}^{\infty} \beta_n \|Sx_n - Sx_{n-1}\| < \infty$.

Theorem 3.4.2 *Let D be a real Hilbert space and let $S, T : D \to D$ satisfy the assumptions of Theorem 3.3.1, including:*

(i) $T(D) \subseteq S(D)$;
(ii) S is injective and weakly continuous;
(iii) $C(S,T) \neq \emptyset$;
(iv) T is Jungck-strictly pseudo-contractive with respect to S;
(v) (S,T) satisfies the Jungck monotonicity condition.
(vi) The control sequence α_n satisfies
$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be generated by the inertial Jungck–Halpern scheme.

Then there exists $p \in C(S,T)$ such that $Sx_n \to Sp$ strongly as $n \to \infty$.

Moreover, if S^{-1} exists and is continuous on $S(D)$, then $x_n \to p$ strongly.

Proof: Let $p \in C(S,T)$ be arbitrary. Then $Tp = Sp$.

From the inertial step,
$$Sy_n - Sp = (Sx_n - Sp) + \beta_n (Sx_n - Sx_{n-1}).$$

Taking norms and using the identity
$$\| a + b \|^2 \leq \| a \|^2 + \| b \|^2 + 2\langle a, b \rangle,$$
We obtain
$$\| Sy_n - Sp \|^2 \leq \| Sx_n - Sp \|^2 + \beta_n^2 \| Sx_n - Sx_{n-1} \|^2 + 2\beta_n \langle Sx_n - Sp, Sx_n - Sx_{n-1} \rangle.$$

Using the summability condition
$$\sum \beta_n \|Sx_n - Sx_{n-1}\| < \infty$$
we deduce that the inertial perturbation is asymptotically negligible, i.e.,
$$\| Sy_n - Sx_n \| \to 0.$$

Now consider the main iteration:
$$Sx_{n+1} = \alpha_n Su + (1 - \alpha_n) Ty_n.$$

Proceeding as in Theorem 3.3.1, we obtain
$$\| Sx_{n+1} - Sp \|^2 \leq \alpha_n \| Su - Sp \|^2 + (1 - \alpha_n) \| Ty_n - Tp \|^2.$$

Applying the Jungck-strict pseudo-contractive condition yields
$$\| Ty_n - Tp \|^2 \leq \| Sy_n - Sp \|^2 + L \| Sy_n - Ty_n \|^2.$$

Since $\| Sy_n - Sx_n \| \to 0$ and $\| Sy_n - Ty_n \| \to 0$, we obtain
$$\| Ty_n - Tp \|^2 = \| Sx_n - Sp \|^2 + o(1).$$

Substituting into the main inequality gives
$$\| Sx_{n+1} - Sp \|^2 \leq (1 - \alpha_n) \| Sx_n - Sp \|^2 + \alpha_n \| Su - Sp \|^2 + o(1).$$

This implies that $\{\| Sx_n - Sp \|\}$ is Fejér monotone up to a vanishing error.

From the estimates established in the preceding steps, there exists a constant

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$$\alpha_n C = \alpha_n \parallel Su - Sp \parallel^2$$

such that

$\parallel Sx_{n+1} - Sp \parallel^2 \leq (1-\alpha_n) \parallel Sx_n - Sp \parallel^2 + \alpha_n C + \delta_n,$ $where\ \delta_n = \beta_n^2 \|Sx_n - Sx_{n-1}\|^2 \geq 0$

The term δ_n represents the higher-order perturbation introduced by the inertial component.

Let $a_n = \|Sx_n - Sp\|^2, \gamma_n = \alpha_n, and\ \sigma_n = \delta_n.$

Then the above inequality becomes

$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n C + \sigma_n.$

By assumption, $\alpha_n \to 0, \sum_{n=0}^{\infty} \alpha_n = \infty,$

and the perturbation satisfies

$\sum_{n=0}^{\infty} \delta_n < \infty,$ we have $\sum_{n=0}^{\infty} \sigma_n < \infty.$

Furthermore, $\lim_{n\to\infty} \sup(C - a_n) \leq 0$

Because every weak cluster point of $\sum_{n=0}^{\infty} \sigma_n < \infty. \{x_n\}$ belongs to $C(S,T)$, and the quasi-Fejér monotonicity established earlier implies that the distance sequence converges to the minimum possible value.

Therefore, all assumptions of Lemma 2.5 are satisfied. Hence, $a_n \to 0, thatis, \|Sx_n - Sp\| \to 0.$

Since S^{-1} is continuous on S(D), $x_n \to p.$

An accelerated two-step Jungck iterative scheme

In this section, we propose an accelerated two-step iterative method for Jungck-strictly pseudo-contractive mappings. The scheme incorporates an intermediate correction step intended to improve convergence performance.

Definition 3.5.1 (Accelerated two-step Jungck iteration)

Let D be a real Hilbert space and let $S,T: D \to D$ be self-mappings. Define $\{x_n\}$ by $Sy_n = (1-\beta_n)Sx_n + \beta_n Tx_n,$

$$Sx_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tx_n, \qquad n \geq 0,$$

where the control sequences satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0;$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty;$

(iii) $0 \leq \beta_n \leq \beta < 1.$

Theorem 3.5.2: *Let D be a real Hilbert space and let $S,T: D \to D$ satisfy:*

(i) $T(D) \subseteq S(D);$

(ii) S is injective and weakly continuous;

(iii) $C(S,T) \neq \emptyset;$

(iv) T is Jungck-strictly pseudo-contractive with respect to $S;$

(v) (S,T) satisfies the Jungck monotonicity condition.

(vi) The control sequence α_n satisfies

$$\lim_{n\to} \alpha_n = 0 \ \text{ and } \ \sum_{n=1}^{\infty} \alpha_n = \infty$$

Then, the accelerated two-step iteration $\{x_n\}$ defined above satisfies

$Sx_n \to Sp$ strongly as $n \to \infty$, for some $p \in C(S,T)$. Moreover, if S^{-1} exists and is continuous on $S(D)$, then $x_n \to p$ strongly.

Proof: Let $p \in C(S,T)$ so that $Tp = Sp.$

From the first step of the iteration in Definition 3.5.1,

$$Sy_n - Sp = (1-\beta_n)(Sx_n - Sp) + \beta_n(Tx_n - Tp).$$

Hence,

$$\parallel Sy_n - Sp \parallel \leq (1-\beta_n) \parallel Sx_n - Sp \parallel + \beta_n \parallel Tx_n - Tp \parallel.$$

Using the Jungck-strict pseudo-contractive condition,

$$\parallel Tx_n - Tp \parallel^2 \leq \parallel Sx_n - Sp \parallel^2 + L \parallel Sx_n - Tx_n \parallel^2,$$

we obtain $\parallel Sy_n - Sp \parallel \leq \parallel Sx_n - Sp \parallel + o(1).$

From the second step of the iteration in Definition 3.5.1,

$$Sx_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tx_n,$$

We deduce $\parallel Sx_{n+1} - Sp \parallel^2 \leq (1-\alpha_n) \parallel Ty_n - Tp \parallel^2 + \alpha_n \parallel Tx_n - Tp \parallel^2.$

Since $Sy_n - Sx_n \to 0$ and $Ty_n - Tx_n \to 0,$

we conclude

$$\parallel Ty_n - Tp \parallel^2 = \parallel Sx_n - Sp \parallel^2 + o(1).$$

Thus, $\parallel Sx_{n+1} - Sp \parallel^2 \leq (1-\alpha_n) \parallel Sx_n - Sp \parallel^2 + \alpha_n C + o(1),$ for some constant $C > 0.$

From the estimates obtained in the previous steps, there exists a constant

$$C = \|Su - Sp\|^2$$

such that

$\parallel Sx_{n+1} - Sp \parallel^2 \leq (1-\alpha_n) \parallel Sx_n - Sp \parallel^2 + \alpha_n C + \delta_n,$ $where\ \delta_n = \beta_n^2 \|Sx_n - Sx_{n-1}\|^2 \geq 0$

δ_n represents the perturbation introduced by the additional averaging step.

Define

$a_n = \parallel Sx_n - Sp \parallel^2, \gamma_n = \alpha_n, \sigma_n = \delta_n.$

Then

$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n C + \sigma_n.$

Since

$\alpha_n \to 0, \sum_{n=0}^{\infty} \alpha_n = \infty, and \sum_{n=0}^{\infty} \delta_n < \infty,$

The perturbation sequence is summable.

Moreover, the same argument as in Theorem 3.4.2 yields $\lim_{n\to\infty} \sup(C - a_n) \leq 0$

Applying Lemma 2.5 yields $a_n \to 0.$

Consequently, $\parallel Sx_n - Sp \parallel \to 0.$

Since S^{-1} is continuous, $x_n \to p.$

Therefore, the accelerated two-step Jungck iteration converges strongly to the coincidence point p.

Remark 3.5.3: The concluding arguments in Theorems 3.3.1, 3.4.2, and 3.5.2 all rely on the same

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application of Xu's convergence lemma. The only distinction lies in the perturbation sequence, δ_n which reflects the additional inertial or averaging terms present in the corresponding iterative scheme. In each iterative scheme, the control sequence $\{\alpha_n\}$ satisfies the two fundamental Halpern conditions

$$\lim_{n\to\infty}\alpha_n = 0 \text{ and } \sum_{n=1}^{\infty}\alpha_n = \infty,$$

while the corresponding perturbation sequence is nonnegative and summable. Consequently, all hypotheses of Xu's convergence lemma are satisfied, yielding strong convergence of the generated sequence to the coincidence point.

Discussion

Numerical illustration and convergence comparison

A nontrivial Jungck-strictly pseudo-contractive example

We construct a nontrivial example of the proposed operator class in the Hilbert space $H = \mathbb{R}^m$ equipped with the standard Euclidean inner product. Let $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite with $\| A \| \le 1$, and choose constants $0 < \delta < \gamma < 1$. Define

$$Sx = (I + \gamma A)x, \qquad Tx = (I + \delta A)x.$$

For arbitrary $u, v \in H$, put $d = u - v$. Then

$$Su - Sv = (I + \gamma A)d, \qquad Tu - Tv = (I + \delta A)d,$$

and

$$(S - T)u - (S - T)v = (\gamma - \delta)Ad.$$

Because A is self-adjoint and positive definite, the spectral theorem gives an orthonormal eigenbasis with eigenvalues $\lambda_i > 0$. Since $0 < \delta < \gamma$, we have

$$(1 + \delta\lambda_i)^2 \le (1 + \gamma\lambda_i)^2$$

for every i. Consequently,

$$\| Tu - Tv \|^2 \le \| Su - Sv \|^2 \le \| Su - Sv \|^2 + L \| (S - T)u - (S - T)v \|^2$$

for every fixed $L \in (0,1)$. Hence, T is Jungck-strictly pseudo-contractive with respect to S. Moreover,

$$\langle Su - Sv, Tu - Tv\rangle = \langle (I + \gamma A)d, (I + \delta A)d\rangle > 0$$

whenever $u \ne v$, so the Jungck monotonicity condition is also satisfied. Since $A \ne 0$ and $\gamma \ne \delta$, this pair is not a trivial identity example.

Verification of the convergence hypotheses

For the numerical experiment, take $H = \mathbb{R}^2$,

$$A = \operatorname{diag}(0.5, 0.3), \qquad \gamma = 0.3, \qquad \delta = 0.1.$$

The two operators are therefore

$$S = I + 0.3A = \operatorname{diag}(1.15, 1.09),$$
$$T = I + 0.1A = \operatorname{diag}(1.05, 1.03).$$

The operator S is bounded, linear, self-adjoint and positive definite. In particular,

$$S^{-1} = \operatorname{diag}\left(\frac{1}{1.15}, \frac{1}{1.09}\right)$$

exists and is continuous. Thus, S is injective and weakly continuous, $S(H) = H$ is closed and convex, and $T(H) \subseteq S(H)$. The coincidence equation is

$$Sx = Tx \quad \Leftrightarrow \quad 0.2Ax = 0.$$

Since A is positive definite, the unique solution is

$$p = (0,0)^T, \qquad C(S,T) = \{p\}.$$

The Jungck-strictly pseudo-contractive and monotonicity conditions follow from a nontrivial Jungck-strictly pseudo-contractive example. Therefore, the mapping assumptions required by Theorem 3.3.1, 3.4.2 and 3.5.2 are satisfied for this example. For the inertial experiment below, the selected sequence $\beta_k = 0.20/(k+2)^2$ is summable; together with boundedness of the image iterates, this also guarantees

$$\sum_{k=0}^{\infty}\beta_k \| Sx_k - Sx_{k-1} \| < \infty.$$

Numerical verification

The numerical implementation follows the displayed iterative schemes exactly. The computational counter is indexed by $k = 0,1,2,\dots$, with

$$\alpha_k = \frac{1}{k+2}, \qquad \beta_k = \frac{0.20}{(k+2)^2}.$$

Hence

$0 < \alpha_k < 1$, $\alpha_k \to 0$, $\sum_{k=0}^{\infty}\alpha_k = \infty$ and $0 \le \beta_k \le 0.05 < 1$.

The common initial approximation and Halpern anchor are

$$x_0 = (2, -1)^T, \qquad u = (0,0)^T.$$

For the inertial scheme, the initialization $x_{-1} = x_0$ was used.

The four implemented algorithms are:

Jungck–Mann comparison scheme

$$Sx_{k+1} = (1 - \alpha_k)Sx_k + \alpha_k Tx_k.$$

Jungck–Halpern scheme

$$Sx_{k+1} = \alpha_k Su + (1 - \alpha_k)Tx_k.$$

Inertial Jungck–Halpern scheme

$$Sy_k = Sx_k + \beta_k(Sx_k - Sx_{k-1}),$$

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$$Sx_{k+1} = \alpha_k Su + (1-\alpha_k)Ty_k.$$

Accelerated two-step Jungck scheme

$$Sy_k = (1-\beta_k)Sx_k + \beta_k Tx_k,$$
$$Sx_{k+1} = (1-\alpha_k)Ty_k + \alpha_k Tx_k.$$

At every update, equations in the image space were converted to iterates by solving the linear system $Sx = z$; the inverse matrix was not formed explicitly. Convergence was assessed using the coincidence residual

$$R_k = \| Sx_k - Tx_k \|_2.$$

The approximation error $E_k = \| x_k - p \|_2$ was also evaluated as an independent accuracy check. An algorithm was terminated when $R_k < 10^{-6}$ or after 10,000 updates, whichever occurred first.

All numerical values were generated with a single Python 3.13.5 script using NumPy 2.3.5, pandas 2.2.3, and Matplotlib 3.10.8. The selected coincidence residuals are reported in Table 1, whereas the complete convergence histories are displayed in Figure 1. Using one implementation for both outputs ensures that the tabulated values, stopping counts, and plotted curves are mutually consistent.

With the common stopping criterion $R_k < 10^{-6}$, the Jungck–Halpern method converged after 112 updates, the inertial Jungck–Halpern method after 111 updates, and the accelerated two-step method after 195 updates. The Jungck–Mann comparison scheme did not reach the tolerance within 10,000 updates; its residual at the final update was approximately 9.9993×10^{-2}.

The results show that the summably damped inertial term gives a modest improvement over the Jungck–Halpern scheme for this example. The accelerated two-step scheme also converges and substantially outperforms the Jungck–Mann baseline, although it is slower than the two Halpern-type schemes for the present parameter selection. The convergence histories in Figure 1 visually confirm the ordering observed in Table 1. Thus, the numerical experiment supports the convergence theory without asserting universal acceleration across all admissible parameter choices.

Table 1. Coincidence residuals at selected iterations for the four iterative schemes.

Iteration k	Jungck–Mann	Jungck–Halpern	Inertial J-H	Accelerated two-step
0	0.208806	0.208806	0.208806	0.208806
1	0.200005	0.095604	0.095604	0.190806
5	0.184474	0.022453	0.021552	0.134044
10	0.175806	0.007949	0.007604	0.086920
20	0.166790	0.001798	0.001718	0.037516
30	0.161526	0.000550	0.000526	0.016953

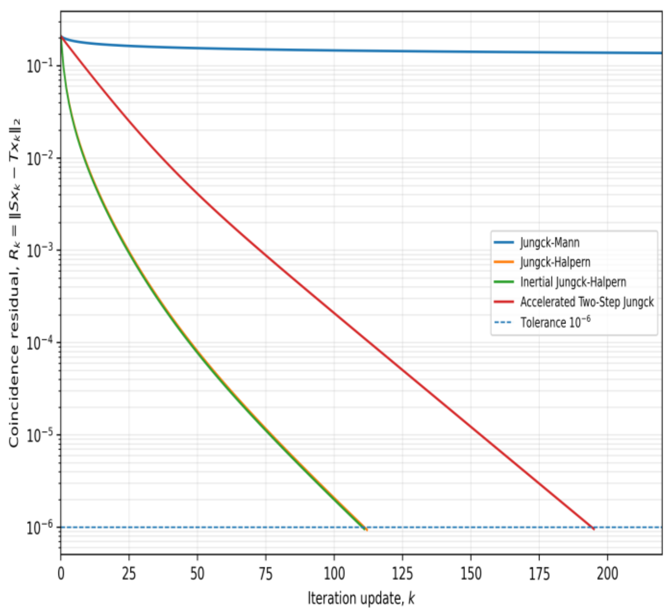

Figure 1. Convergence comparison using the coincidence residual $R_k = \| Sx_k - Tx_k \|_2$ on a logarithmic scale. The dashed line denotes the tolerance 10^{-6}.

Conclusion

In this paper, we introduced a new class of Jungck-strictly pseudo-contractive mappings in real Hilbert spaces and established that it forms a proper subclass of the class of Jungck generalised pseudo-contractive mappings. This structural result demonstrates that the proposed operator class imposes stronger geometric restrictions while retaining a sufficiently broad framework for coincidence point theory

Based on this new operator class, we investigated three iterative approximation methods, namely the Jungck–Halpern iteration, the inertial Jungck–Halpern

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iteration, and an accelerated two-step Jungck iteration. Under suitable assumptions on the mappings and control parameters, we established strong convergence of each algorithm to a coincidence point. The convergence analysis combined quasi-Fejér monotonicity, asymptotic regularity, the demiclosedness principle, and Xu's summability technique.

The numerical experiments reported in Table 1 and illustrated in Figure 1 were generated from the same Python implementation as the displayed iterative formulas. Using the coincidence-residual tolerance 10^{-6}, the inertial Jungck–Halpern method converged after 111 updates, the Jungck–Halpern method after 112 updates, and the accelerated two-step method after 195 updates. The Jungck–Mann comparison scheme did not reach the tolerance within 10,000 updates. These findings support the theoretical convergence conclusions for the representative example and show that the relative speeds of the schemes depend on the selected admissible control sequences.

Despite these contributions, the present work has several limitations. First, the analysis is carried out exclusively in real Hilbert spaces, where the inner-product structure plays a crucial role in establishing the convergence results. Consequently, the current proofs do not immediately extend to general Banach spaces, particularly those lacking uniform convexity or smoothness. Second, the convergence theory is developed under deterministic settings with exact operator evaluations, and therefore does not address stochastic perturbations, computational errors, or inexact iterative implementations. Finally, the numerical study is limited to illustrative finite-dimensional examples and does not include large-scale computational benchmarks or applications arising from practical models.

These limitations suggest several directions for future research. An important extension is developing analogous convergence results in uniformly convex and uniformly smooth Banach spaces, where the absence of an inner product requires fundamentally different analytical techniques based on duality mappings and generalised projection operators. Other promising directions include studying inertial and accelerated Jungck-type algorithms under stochastic or inexact computations, adaptive parameter-selection strategies, variable-metric and viscosity-type modifications, and applications to variational inequality problems, equilibrium problems, monotone inclusion problems, nonlinear integral equations, and nonlinear boundary value problems. Such investigations would considerably broaden the applicability of the proposed framework and further strengthen the connection between Jungck-type iterative methods and contemporary nonlinear analysis.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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