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# Accelerated inertial Jungck iterative scheme for Jungck generalised pseudocontractive and $S$-Lipschitzian mappings in Hilbert spaces

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**ABSTRACT**

In this paper, we introduce an accelerated inertial Jungck iterative framework for approximating common fixed points of Jungck generalised pseudocontractive and $S$-Lipschitzian mappings in real Hilbert spaces. The proposed iteration combines inertial extrapolation, predictor–corrector approximation and viscosity-type stabilisation to improve the convergence behaviour of the classical Jungck–Schaefer and Jungck–Kirk–Mann iterative schemes. Under suitable assumptions on the pair of commuting mappings and admissible control sequences, we establish boundedness, asymptotic regularity, weak convergence and strong convergence of the generated sequence to a common fixed point. Furthermore, we obtain an asymptotic error-majorant comparison theorem for the accelerated core of the proposed method under explicit uniform control conditions. Our results extend and improve several related results in the literature, including those of Olatinwo and Omidire (2021) and related Jungck iterative frameworks.

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## Introduction

The theory of fixed points for nonlinear mappings continues to occupy an important position in nonlinear analysis owing to its wide range of applications in differential equations, optimisation, mathematical physics and engineering sciences. Since Banach's pioneering work [1], several contractive and nonexpansive frameworks have been developed, along with iterative procedures for approximating their fixed points.

Among the classical developments in fixed point theory are the iterative procedures of Mann [2], Krasnosel'skii [3], Schaefer [4] and Kirk [5]. These methods have provided useful tools for the approximation of fixed points of various classes of nonlinear operators and have motivated extensive research into generalised contractive-type mappings and their convergence properties.

In 1976, Jungck [6] introduced fixed point results for commuting mappings and developed iterative frameworks involving pairs of operators. Since then, Jungck-type iterations have been extensively investigated and generalised in different settings [7].

Verma [8] introduced generalised pseudocontractive and Lipschitzian-type mappings and established convergence results for such operators in Hilbert spaces. Motivated by these developments, Olatinwo and Omidire [9] introduced Jungck generalised pseudocontractive and Lipschitzian-type operators and studied the convergence of Jungck–Schaefer and Jungck–Kirk–Mann iterations to the unique common fixed point of the corresponding pair of mappings.



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Although these classical Jungck iterative procedures are mathematically significant, recent developments in nonlinear fixed point theory have shown that inertial and accelerated iterative techniques frequently exhibit improved convergence properties when compared with first-order methods. In particular, inertial extrapolation methods inspired by momentum-type algorithms have proved effective in reducing oscillatory behaviour and improving convergence speed in several nonlinear problems.

Motivated by these observations and by the results of Olatinwo and Omidire [9] and Verma [10], the purpose of this paper is to develop an accelerated inertial Jungck iterative framework for Jungck generalised pseudocontractive and $S$-Lipschitzian mappings in Hilbert spaces.

The contributions of the present paper are summarised as follows:

1. We introduce a new inertial accelerated Jungck iterative scheme incorporating extrapolation and predictor–corrector mechanisms.
2. We establish boundedness and asymptotic regularity of the generated sequence.
3. We prove weak and strong convergence theorems under suitable assumptions.
4. We derive a rigorously qualified comparison result for asymptotic error majorants of the proposed iteration relative to Jungck–Schaefer and Jungck–Kirk–Mann methods.
5. Our results extend and improve several existing convergence results in the Jungck fixed point literature.

The remainder of this paper is organised as follows. In Section 2, we present preliminary definitions and auxiliary results required for the sequel. Section 3 introduces the proposed inertial accelerated Jungck iteration and establishes its convergence properties. Section 4 presents acceleration and comparison results, and the final section provides concluding remarks.

## Preliminaries and definitions

Throughout this paper, let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $D$ be a nonempty closed convex subset of $H$ and let $S,G:D\to D$ be self-mappings. We denote the coincidence point set of $S$ and $G$ by

$$F(S,G)=\{u\in D: Su=Gu\}.$$

**Definition 2.1** ([9]). *Let $D$ be a real Hilbert space and let $S,G:D\to D$ be operators. Then $G$ is called a Jungck generalised pseudocontractive mapping with respect to $S$ if there exists a constant $t>0$ such that, for all $u,v\in D$,*

$$\| Gu-Gv\|^2\le t^2\| Su-Sv\|^2+\| Gu-Gv-t(Su-Sv)\|^2. \quad (1)$$

*Equivalently,*

$$\langle Gu-Gv, Su-Sv\rangle\le t\| Su-Sv\|^2, \qquad u,v\in D. \quad (2)$$

**Remark 2.2**. *Expanding the last term in (1) gives*

$$\| Gu-Gv-t(Su-Sv)\|^2=\| Gu-Gv\|^2- 2t\langle Gu-Gv, Su-Sv\rangle + t^2\| Su-Sv\|^2,$$

*from which (2) follows immediately.*

**Definition 2.3** ([9]). *Let $D$ be a Banach space and let $S,G:D\to D$ satisfy $G(D)\subseteq S(D)$. Then $G$ is called $S$-Lipschitzian if there exists a constant $L>0$ such that*

$$\| Gu-Gv\|\le L\| Su-Sv\|, \qquad u,v\in D. \quad (3)$$

**Definition 2.4**. *Let $S,G:D\to D$ be mappings. Then $S$ and $G$ are said to commute if*

$$SGu=GSu, \qquad u\in D. \quad (4)$$

**Definition 2.5** (Jungck–Schaefer iteration, [9]). *Let $S,G:D\to D$ satisfy $G(D)\subseteq S(D)$. For arbitrary $u_0\in D$, the sequence $\{Su_n\}$ defined by*

$$Su_{n+1}=(1-\lambda)Su_n+\lambda Gu_n, \qquad n\ge 0, \quad (5)$$

*where $\lambda\in(0,1)$, is called the Jungck–Schaefer iterative process.*

**Definition 2.6** (Jungck–Kirk–Mann iteration, [9]). *Let $S,T_i:D\to D$, $i=1,2,\dots,k$, with $T_i(D)\subseteq S(D)$. For arbitrary $u_0\in D$, the sequence $\{Su_n\}$ defined by*

$$Su_{n+1}=\delta_{n,0}Su_n+\sum_{i=1}^{k}\delta_{n,i}T_iu_n, \quad (6)$$

*where $\delta_{n,i}\ge 0$ and*

$$\sum_{i=0}^{k}\delta_{n,i}=1, \qquad \delta_{n,0}\ne 0, \quad (7)$$

*is called the Jungck–Kirk–Mann iterative process. Here the zeroth term is $Su_n$ (equivalently $T_0=S$), which is the form used in the comparison estimate below.*

## Main results

**Definition 3.1** (Inertial Accelerated Jungck Iteration). *Let $D$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $S,G:D\to D$ satisfy $G(D)\subseteq S(D)$. Let $u_0,u_1,f\in D$. The sequence $\{Su_n\}$ is generated by*

$$\begin{cases} w_n &= Su_n+\theta_n(Su_n-Su_{n-1}),\\ Sv_n &= (1-\beta_n)w_n+\beta_nGu_n,\\ Su_{n+1} &= \alpha_nSf+(1-\alpha_n)[(1-\sigma_n)Sv_n+\sigma_nGv_n],\end{cases} \qquad n\ge 1, (8)$$

*where $0\le\theta_n<1$ and $\alpha_n,\beta_n,\sigma_n\in(0,1)$.*

**Remark 3.2** (Well-definedness of the Jungck steps). *The inclusion $G(D)\subseteq S(D)$ does not imply that*



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*$S(D)$ is closed, nor does it by itself guarantee that the inertial point $w_n$ or the convex combination defining $Sv_n$ lies in $S(D)$. To make (8) well-defined without an implicit range assumption, we henceforth set*

$$C := S(D)$$

*and assume that $C$ is a nonempty closed affine subset of $H$. Since affine sets are stable under affine combinations, $Su_n, Su_{n-1} \in C$ implies $w_n \in C$; because $Gu_n \in C$, we also obtain $Sv_n \in C$, and the final convex combination defining $Su_{n+1}$ again lies in $C$. Therefore a preimage $v_n \in D$ and then $u_{n+1} \in D$ exist at every step.*

### Assumption A and reduction to a single operator on $S(D)$

Assume throughout that $C = S(D)$ is a nonempty closed affine subset of $H$, $G(D) \subseteq C$, $F(S,G) \neq \varnothing$, and $G$ is both Jungck generalised pseudocontractive with respect to $S$ and $S$-Lipschitzian with constants $t > 0$ and $L > 0$.
The $S$-Lipschitz condition implies that if $Su = Sv$, then $Gu = Gv$. Consequently the mapping

$$T: C \to C, \qquad T(Su) := Gu, \qquad (9)$$

is well-defined. If $S$ happens to be bijective, then $T = GS^{-1}$ on $C$. Moreover, for $x = Su$ and $y = Sv$,

$$\| Tx - Ty \| \leq L \| x - y \|, \qquad (10)$$

$$\langle Tx - Ty, x - y \rangle \leq t \| x - y \|^2. \qquad (11)$$

Also,

$$z \in \mathrm{Fix}(T) \quad \Leftrightarrow \quad z = Sp = Gp \text{ for some } p \in F(S,G). \quad (12)$$

Writing $z_n = Su_n$, $z_f = Sf$ and $y_n = Sv_n$, the iteration (8) becomes

$$\begin{cases} w_n &= z_n + \theta_n(z_n - z_{n-1}), \\ y_n &= (1-\beta_n)w_n + \beta_n T z_n, \\ z_{n+1} &= \alpha_n z_f + (1-\alpha_n)[(1-\sigma_n)y_n + \sigma_n T y_n]. \end{cases} \qquad (13)$$

This equivalent formulation removes the ambiguity between a coincidence point $p \in D$ and its image $Sp \in C$.

**Lemma 3.3** (Demiclosedness principle for the induced operator). *Let $C$ be a nonempty closed convex subset of a Hilbert space and let $T: C \to C$ be nonexpansive. Then $I - T$ is demiclosed at $0$: if $x_n \rightharpoonup x$ and $\| x_n - Tx_n \| \to 0$, then $x \in \mathrm{Fix}(T)$.*

*Proof.* Since a closed convex subset of a Hilbert space is weakly closed, $x \in C$. Suppose $Tx \neq x$. Hilbert spaces satisfy Opial's condition [11,12], hence

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - Tx \|$$

On the other hand, nonexpansiveness gives

$$\| x_n - Tx \| \leq \| x_n - Tx_n \| + \| Tx_n - Tx \|$$

$$\leq \| x_n - Tx_n \| + \| x_n - x \|.$$

Taking lower limits and using $\| x_n - Tx_n \| \to 0$ contradicts Opial's strict inequality. Hence $Tx = x$. This is the standard demiclosedness principle, here stated for the induced operator $T = GS^{-1}$ when $S^{-1}$ exists, or equivalently for the well-defined quotient operator (9).

**Theorem 3.4** (Weak convergence). *Assume Assumption A and, in addition, $L \leq 1$. Let $\{z_n\} = \{Su_n\}$ be generated by (13). Suppose*

1. $0 \leq \alpha_n < 1$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$;
2. $0 < \underline{\beta} \leq \beta_n \leq \overline{\beta} < 1$;
3. $0 \leq \sigma_n \leq 1$;
4. $r_n := \theta_n \| z_n - z_{n-1} \|$ *satisfies* $\sum_{n=1}^{\infty} r_n < \infty$.

*Then there exists $p^* \in F(S,G)$ such that*

$$Su_n = z_n \rightharpoonup Sp^* = Gp^*.$$

*Proof.* Because $L \leq 1$, (10) shows that $T$ is nonexpansive. For $\lambda \in [0,1]$ define

$$R_\lambda = (1-\lambda)I + \lambda T.$$

If $z \in \mathrm{Fix}(T)$, the Hilbert-space convexity identity and nonexpansiveness yield

$$\| R_\lambda x - z \|^2 \leq \| x - z \|^2 - \lambda(1-\lambda) \| x - Tx \|^2. \quad (14)$$

Let $h_n = \theta_n(z_n - z_{n-1})$, so $\| h_n \| = r_n$. From (13),

$$y_n = R_{\beta_n} z_n + (1-\beta_n)h_n, \qquad q_n := R_{\sigma_n} y_n,$$

$$z_{n+1} = \alpha_n z_f + (1-\alpha_n) q_n.$$

For any $z \in \mathrm{Fix}(T)$, nonexpansiveness of $R_{\beta_n}$ and $R_{\sigma_n}$ gives

$$\| z_{n+1} - z \| \leq \| z_n - z \| + r_n + \alpha_n \| z_f - z \|. \quad (15)$$

Since $\sum r_n < \infty$ and $\sum \alpha_n < \infty$, the standard almost-Fejér sequence lemma implies that $\lim_{n\to\infty} \| z_n - z \|$ exists for every $z \in \mathrm{Fix}(T)$. In particular, $\{z_n\}$ is bounded.
We next establish asymptotic regularity. Let

$$\kappa := \underline{\beta}(1-\overline{\beta}) > 0.$$

Using (14) with $\lambda = \beta_n$, boundedness, and $\| h_n \| = r_n$, there is a constant $M > 0$ such that

$$\| y_n - z \|^2 \leq \| z_n - z \|^2 - \kappa \| z_n - Tz_n \|^2 + M r_n + r_n^2. \quad (16)$$

Since $R_{\sigma_n}$ is nonexpansive and $z_{n+1}$ is a convex combination of $z_f$ and $q_n$,

$$\| z_{n+1} - z \|^2 \leq \| z_n - z \|^2 - (1-\alpha_n)\kappa \| z_n - Tz_n \|^2 + \varepsilon_n, \quad (17)$$

where $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ because $\sum r_n < \infty$ and $\sum \alpha_n < \infty$. Since $\alpha_n \to 0$, summing (17) gives

$$\sum_{n=1}^{\infty} \| z_n - Tz_n \|^2 < \infty, \qquad \text{hence} \qquad \| z_n - Tz_n \| \to 0. \quad (18)$$

Every bounded sequence in a Hilbert space has a weakly convergent subsequence. Let $z_{n_k} \rightharpoonup \bar{z}$. By the demiclosedness lemma and (18), $\bar{z} \in \mathrm{Fix}(T)$.

It remains to prove uniqueness of the weak cluster point. Suppose that $p, q \in \mathrm{Fix}(T)$ are weak



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limits of subsequences $z_{n_k}$ and $z_{m_j}$, respectively, and $p \neq q$. Since the full limits

$$\ell_p := \lim_{n\to\infty} \| z_n - p \|, \qquad \ell_q := \lim_{n\to\infty} \| z_n - q \|$$

exist by (15), Opial's condition applied to $z_{n_k} \rightharpoonup p$ gives $\ell_p < \ell_q$, whereas Opial's condition applied to $z_{m_j} \rightharpoonup q$ gives $\ell_q < \ell_p$, a contradiction. Thus, the weak cluster point is unique, and the entire sequence converges weakly to some $z^* \in \mathrm{Fix}(T)$. Choosing $p^* \in D$ with $Sp^* = z^*$, (12) gives $p^* \in F(S,G)$ and $z^* = Sp^* = Gp^*$.

**Theorem 3.5** (Strong convergence). *Assume Assumption A with* $0 < t < 1$. *Define*

$$\lambda_* := \frac{2(1-t)}{(1+L)^2}. \qquad (19)$$

*Suppose*

1. $\alpha_n \to 0$;
2. $0 < \underline{\beta} \le \beta_n \le \overline{\beta} < \min\{1, \lambda_*\}$;
3. $0 < \underline{\sigma} \le \sigma_n \le \overline{\sigma} < \min\{1, \lambda_*\}$;
4. $r_n = \theta_n \| z_n - z_{n-1} \| \to 0$ *(in particular, condition* $\sum r_n < \infty$ *is sufficient).*

*Then* $F(S,G)$ *has a unique image* $z^* = Sp^* = Gp^*$ *and* $Su_n = z_n \to z^*$ *strongly.*

*Proof.* First, (11) with two fixed points $x, y \in \mathrm{Fix}(T)$ gives

$$\| x-y \|^2 = \langle Tx - Ty, x-y \rangle \le t \| x-y \|^2.$$

Since $t<1$, $\mathrm{Fix}(T)$ is a singleton, say $\{z^*\}$. For $\lambda \in (0, \lambda_*)$ and $R_\lambda = (1-\lambda)I + \lambda T$, we have

$$\begin{aligned} \| R_\lambda x - z^* \|^2 &= \| x - z^* + \lambda(Tx - x) \|^2 \\ &\le [1 - 2\lambda(1-t) + \lambda^2(1+L)^2] \| x - z^* \|^2. \end{aligned}$$

Set

$$q(\lambda) := \sqrt{1 - 2\lambda(1-t) + \lambda^2(1+L)^2}. \qquad (20)$$

The restriction $0 < \lambda < \lambda_*$ gives $0 \le q(\lambda) < 1$. By compactness of the control intervals, there are constants

$$q_\beta := \sup_n q(\beta_n) < 1, \qquad q_\sigma := \sup_n q(\sigma_n) < 1.$$

Using $y_n = R_{\beta_n} z_n + (1-\beta_n) h_n$ and $q_n = R_{\sigma_n} y_n$,

$$\| y_n - z^* \| \le q_\beta \| z_n - z^* \| + r_n,$$

and

$$\| q_n - z^* \| \le q_\sigma q_\beta \| z_n - z^* \| + q_\sigma r_n.$$

Therefore, from the final line of (13),

$$\| z_{n+1} - z^* \| \le q \| z_n - z^* \| + \delta_n, \; q := q_\sigma q_\beta < 1, \qquad (21)$$

where

$$\delta_n := \alpha_n \| z_f - z^* \| + q_\sigma r_n \to 0.$$

A standard contraction-with-vanishing-errors argument applied to (21) yields $\| z_n - z^* \| \to 0$.

**Numerical Example and Simulation**

Take $H = D = \mathbb{R}$ and define

$$Sx = 2x, \qquad Gx = \tfrac{1}{2}x. \qquad (22)$$

Then $S \neq I$ and $G \neq I$, while

$$SGx = S\left(\frac{1}{2}x\right) = x = G(2x) = GSx,$$

so the mappings commute. Moreover, $G(D) = S(D) = \mathbb{R}$ and

$$F(S,G) = \{0\}.$$

For $x, y \in \mathbb{R}$,

$$|Gx - Gy| = \frac{1}{2}|x-y| \le 0.40\,|2x - 2y| = 0.40\,|Sx - Sy|,$$

so $G$ is $S$-Lipschitzian with the admissible constant $L = 0.40$. Also,

$$\begin{aligned} \langle Gx - Gy, Sx - Sy \rangle = (x-y)^2 &\le 0.30\, 4(x-y)^2 \\ &= 0.30\, \| Sx - Sy \|^2, \end{aligned}$$

so the generalised pseudocontractive inequality is satisfied with $t = 0.30$. Consequently,

$$\lambda_* = \frac{2(1-0.30)}{(1+0.40)^2} = 0.714285 \ldots.$$

Choose

$$\alpha_n = \frac{1}{n+1}, \qquad \beta_n = 0.40, \qquad \sigma_n = 0.30,$$
$$\theta_n = \frac{0.20}{n+1},$$

with $u_0 = 5$, $u_1 = 4$ and $f = 0$. Both $\beta_n$ and $\sigma_n$ lie below $\lambda_*$, as required by Theorem 3.5. Since $Su_n = 2u_n$, the iteration is computed from

$$\begin{aligned} w_n &= 2u_n + \theta_n(2u_n - 2u_{n-1}), \\ 2v_n &= (1-\beta_n) w_n + \frac{\beta_n}{2} u_n, \\ 2u_{n+1} &= (1-\alpha_n)\left[(1-\sigma_n)(2v_n) + \frac{\sigma_n}{2} v_n\right]. \end{aligned}$$

The resulting iterates are:

| n | $\lvert u_n \rvert$ |
|---|---|
| 0 | 5.000000000 |
| 1 | 4.000000000 |
| 2 | 1.061750000 |
| 3 | 0.323275750 |
| 4 | 0.118655676 |
| 5 | 0.048451817 |
| 10 | 0.000986520 |
| 15 | 0.000028801 |
| 20 | 0.000000968 |

The iterates converge rapidly to the unique coincidence point 0.

**Theorem 4.1** (Comparison of asymptotic error majorants). *Assume* $0 < L < 1$ *and let* $p \in F(S,G)$. *Suppose the IAJI controls satisfy*

$$0 < \underline{\beta} \le \beta_n < 1, \qquad 0 < \underline{\sigma} \le \sigma_n < 1,$$

*and define*



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$$\overline{c} := \left[1-\underline{\sigma}(1-L)\right]\left[1-\underline{\beta}(1-L)\right] < 1. \tag{23}$$

*For the Jungck–Schaefer method, let*

$$c_S = 1-\lambda+\lambda L.$$

*For the Jungck–Kirk–Mann method, assume, for* $i = 1,\dots,k$,

$$\| T_i u - T_i p \| \le \eta_i \| Su - Sp \|, \qquad 0 \le \eta_i < 1,$$

*and set*

$$\eta_{M,n} := \delta_{n,0} + \textstyle\sum_{i=1}^{k} \delta_{n,i}\,\eta_i, \qquad \eta_M := \sup_{n\ge N} \eta_{M,n} < 1. \tag{24}$$

*Assume*

$$\overline{c} < \eta_M < c_S. \tag{25}$$

*Then the effective IAJI homogeneous contraction coefficient*

$$c_n = (1-\alpha_n)[1-\sigma_n(1-L)][1-\beta_n(1-L)] \tag{26}$$

*satisfies* $c_n \le \overline{c} < \eta_M$ *for every* $n \ge N$.

*If, in addition,* $Sf = Sp$ *and the inertial remainder*

$$r_n = \theta_n \| Su_n - Su_{n-1} \|$$

*is rate-compatible in the sense that*

$$\textstyle\sum_{n=N}^{\infty} \frac{r_n}{\overline{c}^{\,n-N+1}} < \infty, \tag{27}$$

*then the IAJI error possesses an upper majorant of order* $O(\overline{c}^{\,n})$, *whereas the Jungck–Kirk–Mann and Jungck–Schaefer error estimates possess the standard upper majorants* $O(\eta_M^n)$ *and* $O(c_S^n)$, *respectively. Thus, the IAJI bound decays faster in the asymptotic error-majorant sense.*

*Proof.* For $z_n = Su_n$ and $z^* = Sp$, the $S$-Lipschitz inequality gives

$$\| (1-\sigma_n)y + \sigma_n T y - z^* \| \le [1-\sigma_n(1-L)] \| y - z^* \|.$$

Similarly,

$$\| Sv_n - z^* \| \le [1-\beta_n(1-L)] \| z_n - z^* \| + (1-\beta_n) r_n.$$

when $Sf = Sp$, substitution into the final IAJI step yields

$$\| z_{n+1} - z^* \| \le c_n \| z_n - z^* \| + K r_n \tag{28}$$

for a constant $K > 0$. Since $\beta_n \ge \underline{\beta}$ and $\sigma_n \ge \underline{\sigma}$, (26) gives the uniform estimate

$$c_n \le \overline{c} < \eta_M.$$

Iterating (28) and using (27) gives $\| z_n - z^* \| = O(\overline{c}^{\,n})$.

For Jungck–Kirk–Mann,

$$\| Su_{n+1}^M - Sp \| \le \left(\delta_{n,0} + \sum_{i=1}^{k} \delta_{n,i}\,\eta_i\right) \| Su_n^M - Sp \| \le \eta_M \| Su_n^M - Sp \|,$$

while Jungck–Schaefer satisfies

$$\| Su_{n+1}^S - Sp \| \le c_S \| Su_n^S - Sp \|.$$

Hence their standard geometric majorants are $O(\eta_M^n)$ and $O(c_S^n)$. Because $\overline{c} < \eta_M < c_S$, the ratios of the IAJI majorant to either comparison majorant tend to zero.

The conclusion is deliberately stated for error majorants rather than as an unconditional pointwise ordering of the actual errors. The additive inertial term in (28) must be controlled at a compatible rate; condition (27) makes this dependence explicit.

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