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Orthogonal-Based Second Order Hybrid Initial Value Problem Solver

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Abstract
This work focuses on development of an initial value problem solver by employing a new class of orthogonal polynomial, the basis function. We present the recursive formula of the class of polynomials constructed and adopt collocation technique to develop the method. The method was analyzed for its basic properties and findings show that the method is accurate and convergent.
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Introduction

The second order differential equations arise in many important area of physical problems. The difficulties encounter in solving such problems has led to development of numerical methods. To develop such numerical methods, polynomial plays an important role. Notable among the well-known polynomials are the orthogonal polynomials. Orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The first orthogonal polynomials were the Legendre polynomials. Then came the Chebyshev polynomials, the general Jacobi polynomials, the Hermite and the Laguerre polynomials. All these classical orthogonal polynomials play an important role in many applied problems.

Asymptotic formulae for orthogonal polynomials were first discovered bySzegö, (1975). Lanczos (1938) introduced Chebyshev polynomials as trial function. Several researchers have employed these polynomials as trial functions to formulate algorithms (see Shampine and Watts (1969), Tanner (1979), Dahlquist (1979), Jator (2007), Awoyemi (1991)) which are in block forms for solving second order initial value problems directly.

Fatunla (1994) gave a generalization to blockmethods using some definition inmatrix form.Hence, thismotivates us to extend themethod to blockmethod in solving ODEs using a new set of polynomials.

In this work, we shall employ a non-negative weight function to construct a class of orthogonal polynomials which will serve as trial functions to formulate numerical algorithms for the solution of initial value problems.

Construction of Orthogonal Basis Function

We define the orthogonal polynomial of the first kind of degree n over the interval [-1, 1] with respect to weight function $w(x) = (x^2 - 1)^2$ as

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$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \qquad (1)$$

The following requirements are considered:

$$< q_m(x), q_n(x) > = 0, \quad m = 0,1,2,\ldots,n-1 \qquad (2a)$$

For the purpose of constructing the basis function, we adopt the approach discussed extensively in Adeyefa and Adeniyi (2015) and use additional property (the normalization)

$$q_n(1) = 1 \qquad (2b).$$

Using (2), equation (1) yields

$$\left.\begin{aligned}
q_0(x) &= 1\\
q_1(x) &= x\\
q_2(x) &= \frac{1}{6}(7x^2 - 1)\\
q_3(x) &= \frac{1}{2}(3x^3 - x)\\
q_4(x) &= \frac{1}{16}(33x^4 - 18x^2 + 1)\\
q_5(x) &= \frac{1}{48}(143x^5 - 110x^3 + 15x)\\
q_6(x) &= \frac{1}{32}(143x^6 - 143x^4 + 33x^2 - 1)\\
q_7(x) &= \frac{1}{32}(221x^7 - 273x^5 + 91x^3 - 7x)\\
q_8(x) &= \frac{1}{384}(4199x^8 - 6188x^6 + 2730x^4 - 364x^2 + 7)\\
q_9(x) &= \frac{1}{128}(2261x^9 - 3876x^7 + 2142x^5 - 420x^3 + 21x)\\
q_{10}(x) &= \frac{1}{256}(7429x^{10} - 14535x^8 + 9690x^6 - 2550x^4 + 225x^2 - 3)
\end{aligned}\right\} \qquad (3)$$

In the spirit of Golub and Fischer (1992), equation (3) must satisfy three-term recurrence relation

$$c_j p(t) = (t - a_j) p_{j-1}(t) - b_j p_{j-2}(t), j = 1,2,\ldots, p_{-1}(t) = 0,\ p_0(t) \equiv p_0$$

where

b_j, $c_j > 0$ for $j \geq 1$ (b_1 is arbitrary).

$$c_j p(t) = (n+5)P_{n+1}(x),\ (t - a_j)p_{j-1}(t) = (2n+5)xP_n(x),\ b_j p_{j-2}(t) = nP_{n-1}(x), n = 1,2,\ldots$$

The recursive formula for these orthogonal polynomials is therefore given as

$$P_{n+1}(x) = \frac{1}{n+5}\left[(2n+5)xP_n(x) - nP_{n-1}(x)\right],\ n \geq 1,\ P_0(x) = 1,\ P_1(x) = x$$

This relation, along with the two polynomials $P_0(x)$ and $P_1(x)$, allows the new set of polynomials to be generated recursively.

In what immediately follows, we shall develop an algorithm to integrate second order differential equations where polynomials $q_n(x)$ shall be employed as basis function. Thereafter, the analysis of the method for convergence and implementation of the method through some test problems shall be presented. Finally, conclusion shall be made.

Development of the Method

In this section, our aim is to derive a continuous scheme from which a set of block formula is developed. To make this happen, we shall seek an approximant

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \qquad (4)$$

to obtain the solution of second order initial value problems in ordinary differential equations. Transforming $q_n(x)$ to interval [0, 1], we have $x = \frac{2X - 2x_n - ph}{ph}$, where p varies as the method to be developed. In this case, p = 3, s and k in (4) are points of interpolation and collocation respectively. The procedure involves interpolating (4) at points s = $0, \frac{1}{3}$ and collocating the second derivative of (4) at points k = $0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3$.

The a_r, r = 0(1)9 from the resulting system of equations are obtained as

$$a_r = \alpha_0 y_n + \alpha_{\frac{1}{3}} y_{n+\frac{1}{3}} + h^2 \sum_{j=0}^{3} \beta_j f_{n+j} + h^2 \beta_i f_{n+i},\ r = 0(1)9,\ i = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \qquad (5)$$

Substituting (5) into (4) yields the continuous implicit method

$$y(x) = \alpha_0 y_n + \alpha_{\frac{1}{3}} y_{n+\frac{1}{3}} + h^2 \sum_{j=0}^{3} \beta_j f_{n+j} + h^2 \beta_i f_{n+i},\ i = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \qquad (6)$$

Evaluating equation (6) at x = x_{n+m}, m= $\frac{1}{5}, \frac{1}{4}, \frac{1}{2}, 1, 2, 3$ yields the discrete equations

$$y_{n+m} = \alpha_0 y_n + \alpha_{\frac{1}{3}} y_{n+\frac{1}{3}} + h^2 \sum_{j=0}^{3} \beta_j f_{n+j} + h^2 \beta_i f_{n+i},\ i = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \qquad (7)$$

whose values of α *and* β are given in Table 1.

To develop the block method from the continuous scheme, we adopt the general block formula proposed in Shampine and Watts (1969) in the normalized form given as

$$A^{(0)}Y_m = ey_m + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \qquad (8)$$

Evaluating the first derivative of (6) at x = x_{n+j}, j = $0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3$, substituting the resulting equations and equation (7) into (8) and solving simultaneously gives a block formulae represented as Equation (9) is our desired block method of which its basic properties shall be discussed in the next section.

$$\left.\begin{aligned} y_{n+k} &= \alpha_0 y_n + \psi_0 h y_n' + h^2 \sum_{j=0}^{3} \beta_j f_{n+j} + h^2 \beta_i f_{n+i},\ k = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3,\ i = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \\ y_{n+k}' &= \psi_0 y_n' + h \sum_{j=0}^{3} \beta_j f_{n+j} + h\beta_i f_{n+i},\ k = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3,\ i = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \end{aligned}\right\} \quad (9)$$

Analysis of the Method

Order and Error Constant

Following Henrici (1962), the approach adopted in Fatunla (1991, 1994) and Lambert (1973), we define the local truncation error associated with equation (7) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h^2 \beta_j f(x_n + jh)\right] \quad (10$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (10) in Taylor series about the point x, we obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_{p+3} h^{p+3} y^{p+3}(x)$$

where the C_0 , C_1 , C_2 , C_p , … are obtained as

$$C_0 = \sum_{j=0}^{k} \alpha_j, \quad C_1 = \sum_{j=1}^{k} j\alpha_j, \quad C_2 = \frac{1}{2!}\sum_{j=1}^{k} j^2 \alpha_j,$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)\sum_{j=1}^{k} \beta_j j^{q-3}\right]$$

According to Lambert (1973), equations (7) is of order p if

$C_0 = C_1 = C_2 = \ldots C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$

The $C_{p+2} \neq 0$ is called the error constant and $C_{p+2} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.

Thus, equations (7) is of order 8 with the error constants

$$C_{p+2} = \left[7.76\times10^{-11}, 4.89\times10^{-11}, -5.25\times10^{-11}, -2.59\times10^{-8}, 7.22\times10^{-6}, -1.22\times10^{-4}\right]^T$$

Zero Stability of the Method

According to Lambert (1973), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (9) in vector notation form of column vectors $e = (e_1 \ldots e_r)^T$, $d = (d_1 \ldots d_r)^T$, $y_m = (y_{n+1} \cdots y_{n+r})^T$, $F(y_m) = (f_{n+1} \cdots f_{n+r})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, equation (9) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hdf_n \quad (11)$$

where h is a fixed mesh size within a block.

Hence, based on the definition above, the scheme is zero stable.

Consistency of the Method

According to Lambert (1973), a linear multistep method is said to be consistent if it has order at least one. Owing to this definition, equations (7) and (9) are consistent.

Convergency of the Method

According to the theorem of Dahlquist, the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

4.5 Numerical Experiment

Three test problems are considered to demonstrate the accuracy of the method.

Problem 1:

$$y'' = -1001y' - 1000y,\ y(0) = 1,\ y'(0) = -1,\ h = 0.05$$

$$Exact\ Solution: y(x) = \exp(-x)$$

Problem 2:

$$y'' = -\frac{6}{x}y' - \frac{4}{x^2}y,\ y(1) = 1,\ y'(1) = 1,\ h = \frac{0.1}{32}$$

$$Exact\ Solution: y(x) = \frac{5}{3x} - \frac{2}{3x^4}$$

Problem 3: We consider Vander pol's Oscillator Problem

$$y'' = 2\cos x - \cos^3 x - y' - y - y^2 y',\ y(0) = 0,\ y'(0) = 1,\ h = 0.1$$

whose exact solution is y(x) = sinx

Table 1: Tabular Representation of Discrete Equations

	$y_{n+\frac{1}{5}}$	$y_{n+\frac{1}{4}}$	$y_{n+\frac{1}{2}}$	y_{n+1}	y_{n+2}	y_{n+3}
f_n	$\frac{-559066591}{645848437500}$	$\frac{-146852003}{270888468480}$	$\frac{4222913}{4232632320}$	$\frac{258407}{8266860}$	$-\frac{6193885}{1653372}$	$\frac{23653037}{688905}$
$f_{n+\frac{1}{5}}$	$-\frac{128106749}{5208121800}$	$\frac{-105050234375}{6826389405696}$	$\frac{15252734375}{426649337856}$	$-\frac{210859375}{208324872}$	$\frac{111592890625}{833299488}$	$-\frac{19658828125}{17360406}$
$f_{n+\frac{1}{4}}$	$\frac{238680160256}{12432582421875}$	$\frac{208524331}{20369543040}$	$-\frac{4832648}{159137055}$	$\frac{328042496}{159137055}$	$-\frac{688990208}{2893401}$	$\frac{104746409984}{53045685}$
$f_{n+\frac{1}{3}}$	$-\frac{91280227}{11812500000}$	$-\frac{5095663}{990904320}$	$\frac{1045799}{30965760}$	$-\frac{36593}{30240}$	$\frac{1580795}{12096}$	$-\frac{2625347}{2520}$
$f_{n+\frac{1}{2}}$	$\frac{1617428144}{2421931640625}$	$\frac{2159021}{5079158784}$	$\frac{299791}{198404640}$	$\frac{2781328}{6200145}$	$-\frac{28511696}{1240029}$	$\frac{355815616}{2066715}$
f_{n+1}	$-\frac{5594563}{645848437500}$	$\frac{-1483733}{270888468480}$	$\frac{103099}{33861058560}$	$\frac{119087}{8266860}$	$\frac{24778813}{13226976}$	$-\frac{2953843}{688905}$
f_{n+2}	$\frac{21226229}{203442257812500}$	$\frac{1124087}{17065973514240}$	$\frac{-3103}{53331167232}$	$-\frac{28073}{520812180}$	$\frac{5553811}{104162436}$	$\frac{11209049}{8680203}$
f_{n+3}	$\frac{-6957967}{1989213187500000}$	$\frac{-368323}{16686729633680}$	$\frac{2203}{1042920603648}$	$\frac{7723}{5092385760}$	$\frac{-123143}{185177664}$	$\frac{4323101}{84873096}$

Table 2: Numerical Results of Problem 1

x	Exact	New Method	Error	Error in [2]
0.05	0.95122942450071400909	0.95122942450071076950	3.23959e-15	2.05e-11
0.1	0.90483741803595957316	0.90483741803594163261	1.794055e-14	4.39e-11
0.15	0.86070797642505780723	0.86070797642498871168	6.909555e-14	6.55e-11
0.2	0.81873075307798185867	0.81873075307774470455	2.3715412e-13	8.38e-11
0.25	0.77880078307140486825	0.77880078307062386240	7.8100585e-13	9.86e-11
0.03	0.74081822068171786607	0.74081822067918416587	2.5337002e-12	1.10e-10
0.35	0.70468808971871343435	0.70468808971053769036	8.17574399e-12	1.19e-10
0.4	0.67032004603563930074	0.67032004600930714374	2.6332157e-11	1.24e-10
0.45	0.63762815162177329314	0.63762815153701783346	8.475545968e-11	1.28e-10
0.5	0.60653065971263342360	0.60653065943988919902	2.7274422458e-10	1.30e-10

Table 3: Numerical Results of Problem 2

X	Exact	New Method	Error	Error in [4]
0.003125	1.00307652585769622630	1.00307652585769623090	4.6e-18	3.8354 E-05
0.00625	1.00605750308351628300	1.00605750308351632460	4.16e-17	7.5004E-05
0.009375	1.00894499508883757910	1.00894499508883768810	1.09e-16	1.0592 E-04
0.0125	1.01174101816798852880	1.01174101816798873410	2.053e-16	1.35476 E-04
0.015625	1.01444754268641387440	1.01444754268641420320	3.288e-16	1.55567E-04
0.01875	1.01706649423567260840	1.01706649423567308660	4.782e-16	1.86372E-04
0.021875	1.01959975475628759200	1.01959975475628824410	6.521e-16	1.96055E-04
0.025	1.02204916362943174130	1.02204916362943259040	8.491e-16	2.21045E-04
0.028125	1.02441651873840268050	1.02441651873840374840	1.0679e-15	2.05628E-04
0.03125	1.02670357750080598400	1.02670357750080729130	1.3073e-15	2.77908E-04

Table 4: Numerical Results of Problem 3

X	Exact	New Method	Error
0.1	0.09983341664682815231	0.09983341664641143268	4.16719627e-13
0.2	0.19866933079506121546	0.19866933079151260797	3.54860749e-12
0.3	0.29552020666133957511	0.29552020665229235391	9.0472212e-12
0.4	0.38941834230865049167	0.38941834229214808125	1.650241042e-11
0.5	0.47942553860420300027	0.47942553857875939095	2.544360932e-11
0.6	0.56464247339503535720	0.56464247335967945648	3.535590072e-11
0.7	0.64421768723769105367	0.64421768719198266396	4.570838971e-11
0.8	0.71735609089952276163	0.71735609084353295021	5.598981142e-11
0.9	0.78332690962748338846	0.78332690956173874562	6.574464284e-11
1.0	0.84147098480789650665	0.84147098473329359276	7.460291389e-11

Discussion of Results

Tables 2, 3 and 4 give the numerical results for problems 1, 2 and 3. The superiority of our method is established numerically in Tables 2 and 3 as it compared favourably well with existing methods.

The efficiency of the method, which we further implement on Vander pol's Oscillator Problem, is seen in Table 4 as it reproduces the exact solutions with small error values.

Conclusion

Formulation of initial value problem solver has been developed using a new class orthogonal polynomials with recursive formula. Three test problems have been considered to show the efficiency and accuracy of the method.

Tables 2, 3 and 4 display the accuracy and comparison of the numerical results of the method with existing methods. The method is desirability as its superiority has been established by the numerical results.

We hope to extend the scope of this study to partial differential equation in our future paper.

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